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A path enumeration approach for the analysis of critical activities in fuzzy networks

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ABSTRACT

This paper addresses the problem of determining the degree of possible and necessary criticality of activities as well as determining paths in networks that have fuzzy activity durations. In such networks, activities and paths are reported in a fuzzy representation as being critical, with certain degrees of possibility and necessity, instead of being declared critical or not in a binary way. Although the problem of computing the possibility and necessity degrees of criticality for paths have been investigated in the literature, those problems for activities have not yet been addressed. Herein, an efficient algorithm that relies on a path enumeration technique is proposed to compute the possibility degrees of criticality of activities. Additionally, the proposed algorithm computes paths with maximum and minimum degrees for the necessity of criticality, which correspond to activities. Real-world project networks were used to evaluate the performance of the algorithm.

1. Introduction

The Critical Path Method (CPM) [25] is a network-based method that is designed to assist in the planning, scheduling and control of real-world projects. When the durations of activities are fully known, a CPM provides a minimal project duration and identifies the critical activities and paths. However, precise information about the durations of the activities is seldom available in the real world (the majority of research in this area has assumed fixed crisp information, e.g., in [10]). In fact, uncertainty is an attribute of information, according to [47,48]. For this reason, the Program Evaluation and Review Technique (PERT) [32] and Monte Carlo simulations (e.g., [29,34]) based on Probability Theory have been developed.

Fuzzy Logic, an approach toward measuring imprecision in estimation, may be preferred to Probability Theory when capturing activity duration uncertainty in situations where past data are either unavailable or not relevant [30]. Shipley et al. [35], Lootsma [31] and Herroelen and Leus [22] have compared these approaches. The dominant problems in networks with fuzzy activity durations are determining the possible values of the latest starting times and determining the floats of the activities. The possible values of the earliest starting times can be computed by means of a forward recursion procedure that is comparable to the procedure that is used in traditional CPM [3]. Unfortunately, the backward recursion that is derived from the classical CPM is, indeed, not reliable if the durations are described by means of fuzzy numbers. In fact, the backward recursion takes the imprecision of some durations into account twice [15]. Some studies, e.g., [17,20,33], have replaced crisp operators by their fuzzy counterparts, while others, e.g., [21,24], have proposed a backward recursion that relies on the 'optimistic' fuzzy subtraction and provides good results for specific networks. Zielinski [49] determined the possible values of the latest starting times of the activities.

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Dubois et al. [14] proposed an algorithm that is based on path enumeration to compute optimal intervals for the latest starting times and floats. Fortin et al. [19] introduced a solution to the problem of finding the maximal floats of activities, whereas Yakhchali and Ghodsypour [40] proposed a hybrid genetic algorithm for the problem of finding the minimal floats of activities.

Recent research has been directed at relaxing both the deterministic activity durations and the traditional finish-start precedence relations proposed by CPM. Yakhchali and Ghodsypour [41] solved the problem of determining the possible value of the latest starting time of an activity in networks with generalized precedence relations (see [12,18]) and imprecise durations. They suggested an algorithm for computing both the possible values of the latest starting times and the floats of all of the activities [42] and discussed the criticality of the paths in such networks [43]. Instead of the traditional objective, minimizing the make-span of the project, Yakhchali et al. [45] investigated a project scheduling problem that had irregular starting time costs in networks that had imprecise activity durations.

In the fuzzy approach, the analysis of critical paths and activities constitutes another important and challenging problem. Chen and Hsueh [8] and Chen [7] proposed an approach that was based on the extension principle and linear programming formulation for critical path analysis for a network with fuzzy activity durations. Chen and Huang [9] combined fuzzy set theory with traditional methods to compute the critical degrees of the activities and paths. Liberatore [30] presented an approach for fuzzy critical path analysis that was consistent with the extension principle. Xu et al. [36] presented a new fuzzy chance-constrained programming model to find the solution for multi-project and multi-item investment combinations in pertinent problems.

Zielinski determined the possible values of the latest starting times [49]. The floats of activities cannot be computed from those fuzzy numbers that contain the earliest and latest starting times. Thus, Fortin et al. [19] presented a solution to the problem of finding the maximal float of an activity in networks with interval durations, so that the right shape function of a float can be evaluated. As far as the determination of fuzzy floats is concerned, the problem of determining the left shape of the float membership function is strongly *NP*-Hard [5]. Therefore, calculating the possibility degree of criticality that corresponds to a given activity remains ad hoc. The present study aims at proposing an approach toward solving this problem. Herein, an algorithm is introduced that proves to be capable of evaluating the paths that have the maximum necessity degrees of criticality. This improved algorithm progresses further into evaluating lower bounds that are associated with activities' necessary criticality degrees. A verification of the reliability of this method has been accomplished via real-world problems, and the performance of the method has been confirmed. Furthermore, the present study focuses on the activities counterpart of the current trend of studies, which concentrate on the evaluation of the degrees of criticality of only one activity.

The present paper is organized as follows: after recalling the definitions of the possibility and necessity degrees of criticality that correspond to the activities and paths in section 2, linear programming formulations in activity-on-node networks under the special assumption of fuzzy activity durations for membership functions are presented to evaluate the path degrees of criticality in section 3. By means of the path degrees of criticality, section 4 introduces an efficient algorithm that is based on path enumeration and that computes the possibility degrees of the criticality of the activities as well as the lower bounds for their necessity degrees of criticality.

2. Project networks with fuzzy activity durations

A network $G = \{V, E\}$ ($|V| = n, |E| = m$), which is a project activity-on-node (AON) model, is given. V is the set of nodes (activities), and E is the set of arcs (precedence relations). The following reasons explain why an AON network is chosen:

- A problem with activity-on-arc (AOA) networks is the minimization of the number of dummy activities. Kirshnamoorty and Deo [28] showed that minimizing the number of dummy activities in an AOA network is *NP*-Hard.
- A significant drawback of an AOA network is having several different networks that describe the same project. In contrast, the AON representation is unique. As a result, there may be several free floats for the same activity, depending on the networks. Several authors attempted to cope with this problem, e.g., Cohen and Sadeh [11] proposed an algorithm that generated a unique AOA network from a list of precedence constraints; this algorithm does not minimize the number of dummy activities.
- In addition, precedence relation information, assembled by any information-gathering technique, can be utilized to draw AON networks directly; however, several algorithms have been developed to generate an AOA network from a given AON network, e.g., Kamburowski et al. [23].

The network G is a directed, connected, acyclic graph that includes two virtual activities with a null duration: '1', the first activity and 'n', the last activity. The '1' precedes all of the activities of the network, and each of them precedes 'n'. Dummy activities are needed only to satisfy the requirement that the network possesses only one initial and one terminal node. Assume, without loss of generality, that the activities are topologically numbered such that an arc always leads from a node with a smaller number to a node with a higher number.

In the network G, the set of immediate successors of an activity j, $j \in V$, is denoted by $Succ(j)$, $Succ(j) = \{k | (j,k) \in E\}$. The set of all paths in G from 1 to n will appear in P. $P(k)$ represents the set of all of the paths from P that contain the given activity k,

$P(k) = \{p|p \in P \text{ and } k \in p\}$. The notion of a fuzzy number, which is helpful in developing the formulations, is recalled before moving onto the basic considerations.

2.1. Fuzzy numbers

A fuzzy number, $\widetilde{A}$, is a normal convex fuzzy set in the space of real numbers with an upper semi-continuous membership function, $\mu_{\widetilde{A}}$. A fuzzy set is convex if and only if its membership function is quasiconcave, i.e., it fulfills the condition $\mu_{\widetilde{A}}(z) \geqslant \min\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{A}}(y)\}$ for each *x, y, z* such that $z \in [x,y]$. The following interval, formula (1), is called the $\lambda - cut$ of the fuzzy number $\widetilde{A}$.

$$\widetilde{A}^{\lambda} = [\underline{a}^{\lambda}, \bar{a}^{\lambda}] = \{x \in \Re | \mu_{\widetilde{A}}(x) \geqslant \lambda\}, \quad \text{for } \lambda \in (0,1] \tag{1}$$

A fuzzy number $\widetilde{A}$ is called a fuzzy number of the *L–R type* if its membership function has the following form [16]:

$$\mu_{\widetilde{A}}(x) = \begin{cases} 1 & \text{for } x = [\underline{a}, \bar{a}], \\ L\left(\frac{\underline{a}-x}{\alpha_A}\right), & \text{for } x < \underline{a}, \\ R\left(\frac{x-\bar{a}}{\beta_A}\right), & \text{for } x > \bar{a}, \end{cases} \tag{2}$$

where *L* and *R* are continuous non-increasing functions, which are defined on $[0,+\infty)$ and are strictly decreasing to zero in those subintervals of $[0,+\infty)$ in which they are positive; they satisfy the conditions $L(0) = R(0) = 1$. The parameters α_A and β_A are non-negative real numbers. The fuzzy number of the *L–R type* is denoted by $A = (\underline{a}, \bar{a}, \alpha_A, \beta_A)_{L-R}$ [16]. A fuzzy number $\widetilde{A}$of the *L–R type* $\lambda - cut, \widetilde{A}^{\lambda}, \alpha \in (0,1]$, has the following form:

$$\widetilde{A}^{\lambda} = [\underline{a} - L^{-1}(\lambda)\alpha_A, \bar{a} + R^{-1}(\lambda)\beta_A] \tag{3}$$

where L^{-1} (similarlyR^{-1}) denotes the reverse function to *L* in the part of its domain in which it is positive.

2.2. Fuzzy activity durations

Activity durations are determined by means of fuzzy numbers. Fuzzy numbers express uncertainty connected with the ill-known activity durations that are modeled by these numbers, which generate possibility distributions (see [46]) for the sets of values that contain the unknown durations [13]. The fuzzy number $\tilde{d}_i$ imprecisely determines a duration time of the activity *i*, $i \in V$. The membership function $\mu_{\tilde{d}_i}$ generates a possibility distribution for the duration time of the activity *i*, $i \in V$. It would be accurate to say that the ascertainment of the form "v_i is $\tilde{d}_i$", where v_i is a variable and $\tilde{d}_i$ is a fuzzy number, generates the possibility distribution of v_i, according to the following formula [15]:

$$Poss(v_i = x) = \mu_{\tilde{d}_i}(x), \quad x \in \Re_+ \tag{4}$$

G^{λ} denotes the $\lambda - cut$ of the network, *G*. This definition implies that, for the network *G* with interval activity durations, $\tilde{d}_i^{\lambda} = [\underline{d}_i^{\lambda}, \bar{d}_i^{\lambda}],\ i \in V$.

2.3. The notation of configurations

The notation of the *configurations* denoted by Ω has been defined by Buckley [1] to relate the fuzzy case to the deterministic case of classical PERT/CPM problems. A configuration is a tuple $\Omega\ (d_1,d_2,\ldots,d_n)$ of activity durations such that $d_i \in \Re_+,\ i \in V$. For a configuration Ω, $d_i(\Omega)$ denotes the duration of the activity *i*. $s_i^e(\Omega),\ s_i^l(\Omega)$ and $f_i(\Omega)$ denote the earliest starting time, the latest starting time and the total float of the activity '*i*' in the configuration, Ω, respectively. The pessimistic configuration induced by *p*, $p \in P$, in G^{λ}, denoted by $\overline{\Omega}_p^{\lambda}$, is a configuration such that the following occurs:

$$d_i\left(\overline{\Omega}_p^{\lambda}\right) = \begin{cases} \bar{d}_i^{\lambda} & \text{for } i \in p \\ \underline{d}_i^{\lambda} & \text{for } i \notin p \end{cases} \tag{5}$$

Analogously, $\underline{\Omega}_p^{\lambda}$, called the optimistic configuration induced by *p*, $p \in P$, in G^{λ}, is defined as formula (6):

$$d_i\left(\underline{\Omega}_p^{\lambda}\right) = \begin{cases} \underline{d}_i^{\lambda} & \text{for } i \in p \\ \bar{d}_i^{\lambda} & \text{for } i \notin p \end{cases} \tag{6}$$

The (joint) possibility distribution over the configurations, denoted by $\pi(\Omega)$, is determined by the following formula:

$$\pi(\Omega) = \min_{i \in V} \mu_{\tilde{d}_i}(d_i), \quad \Omega \in \Re_+^n \tag{7}$$

Intuitively, if the activity durations are expressed by fuzzy numbers, then the earliest and latest starting times and floats of activities become fuzzy as well. Thus, the possibility distribution for the earliest and latest starting times and the float of an activity *k* is defined as follows [15]:

$$\mu_{\tilde{S}_k^e}(x) = Poss(s_k^e = x) = \sup_{\Omega: x = s_k^e(\Omega)} \pi(\Omega), \quad x \in \Re_+ \tag{8}$$

$$\mu_{\tilde{S}_k^l}(x) = Poss(s_k^l = x) = \sup_{\Omega: x = s_k^l(\Omega)} \pi(\Omega), \quad x \in \Re_+ \tag{9}$$

$$\mu_{\tilde{F}_k}(x) = Poss(f_k = x) = \sup_{\Omega: x = f_k(\Omega)} \pi(\Omega), \quad x \in \Re_+ \tag{10}$$

The possible values of the earliest starting times of the activities are easily determined by means of a forward recursion procedure that is comparable to the procedure used in the traditional CPM [3]. Zielinski determined the possible values of the latest starting times [49] (see also [39]). The floats of the activities cannot be computed from the fuzzy numbers that contain the earliest and latest starting times. Thus, Fortin et al. [19] provided a solution to the problem of finding the maximal float of an activity in networks with interval durations, so that the correct shape function of $\mu_{\tilde{F}_k}$ can be calculated. With respect to the determination of the fuzzy float $\tilde{F}_k$, the problem of determining the left shape function of $\mu_{\tilde{F}_k}$ is strongly *NP*-Hard [5]. This problem remains *NP*-Hard even in a network that is restricted to be planar [6]. Thus, the problem of calculating the possibility degree of criticality of a given activity remains ad hoc. In the following, a novel solution for this problem is proposed.

3. Criticality in networks with fuzzy activity durations

In traditional CPM, an activity is critical if and only if its earliest and latest starting times are equal; furthermore, critical activities form critical paths that are the longest paths in a given network with crisp activity durations. The criticality concept in networks with fuzzy activity durations is a more realistic approach than the traditional approaches. The concept of criticality in networks with fuzzy activity durations is discussed in this section.

3.1. The degree of possibility that a path is critical

The possibility that a path is critical is defined as the following:

$$Poss(p \text{ is critical}) = \underset{\Omega: p \text{ is critical in } \Omega}{sup} \pi(\Omega) \tag{11}$$

In the literature, two methods have been proposed for computing the degree of possibility that a path is critical. The first method, which is adapted to fuzzy activity durations given in a general form, is based on the idea of a bisection. The second method is based on linear programming. In this method, it is assumed that each fuzzy activity duration $\tilde{d}_i,\ i \in V$, is given by means of a fuzzy number of the *L*–*R* type in which the left shape function L_i is equal to the right shape function R_i, and additionally, the left shape function $L_i$is the same for all of the activity durations, $L = L_i = R_i$, $\forall i \in V$. A linear programming model for this problem in the activity-on-arc (AOA) convention is different from that for the same problem in the activity-on-node (AON) convention. Thus, a different linear programming model for this problem is proposed in this paper. The difference between the linear programming models in AOA and AON will be discussed further below.

Chanas and Zielinski [4] showed that a path, $p \in P$, is possibly critical in G^λ if and only if the path is a critical path in $\overline{\Omega}_p^\lambda$ in the actual sense. The following model of linear equalities and inequalities ascertains whether the given path, p, is possibly critical in network G^λ for a fixed $\lambda \in (0,1]$.

$$\begin{aligned} & s_j - s_i - \bar{d}_i^\lambda = 0 \quad i \in P, \quad j \in P, \quad j \in Succ(i), \\ & s_j - s_i - \bar{d}_i^\lambda \geqslant 0 \quad i \in P, \quad j \notin P, \quad j \in Succ(i), \\ & s_j - s_i - \underline{d}_i^\lambda \geqslant 0 \quad i \notin P, \quad j \notin P, \quad j \in Succ(i), \\ & s_1 = 0, \\ & s_i \geqslant 0, \quad \forall i \in V \end{aligned} \tag{12}$$

where the variables s_i represent the earliest starting times of $i \in V$.

Note that the network must possess only one initial and one terminal node to satisfy the above model requirement. The following formula states the relation between the above model and the degree of possibility that the given path is critical [4].

$$Poss(p \text{ is critical}) = \sup\{\lambda | p \text{ is possibly critical in } G^\lambda\} \tag{13}$$

Thus, the following programming model computes the path's possibility degree of criticality.

$$\begin{aligned} & Maximize \left\{ \lambda - \sum_{i \in V} s_i \right\}, \\ & s_j - s_i - \bar{d}_i^\lambda = 0 \quad i \in P, \quad j \in P, \quad j \in Succ(i), \\ & s_j - s_i - \bar{d}_i^\lambda \geqslant 0 \quad i \in P, \quad j \notin P, \quad j \in Succ(i), \\ & s_j - s_i - \underline{d}_i^\lambda \geqslant 0 \quad i \notin P, \quad j \notin P, \quad j \in Succ(i), \\ & 0 \leqslant \lambda \leqslant 1, \\ & s_1 = 0, \\ & s_i \geqslant 0, \quad \forall i \in V, \end{aligned} \tag{14}$$

The objective function is the significant difference between the model for the activity-on-arc (AOA) convention and the model for the activity-on-node (AON) convention. If the AOA convention is used, then the objective function of the model must maximize, which would be sufficient. However, in the AON convention, this strategy leads to false results. Based on formula (13), the objective function of the model maximizes. If only the *Maximize*$\{\lambda\}$ term is used as the objective function, then the results of the model are not valid because the model gives only the correct earliest starting times for activities that belong to the given path. The term *minimize*$\{\sum_{i\in V} s_i\}$ in the objective function enables the model to compute the value of the earliest starting times of all of the activities correctly. Hence, the earliest starting times of the activities are corollaries of solving the model.

To solve the above model, assume that the fuzzy activity durations, $\tilde{d}_i,\ i \in V$, are given by means of fuzzy numbers of the same L–L type, $\tilde{d}_i = (\underline{d}_i, \bar{d}_i, \alpha_i, \beta_i)_{L-L}$. Based on the $\lambda - cut$ of L–L type fuzzy numbers, evaluated via formula (3), the above model is transferred into the following linear programming problem. In fact, is replaced by $\bar{d}_i$ and is superseded by $\underline{d}_i$.

$$
\begin{aligned}
&Minimize \left\{\theta + \sum_{i\in V} s_i\right\},\\
&s_j - s_i - \bar{d}_i - \beta_i\theta = 0 \quad i \in P, \quad j \in P, \quad j \in Succ(i),\\
&s_j - s_i - \bar{d}_i - \beta_i\theta \geqslant 0 \quad i \in P, \quad j \notin P, \quad j \in Succ(i),\\
&s_j - s_i - \underline{d}_i + \alpha_i\theta \geqslant 0 \quad i \notin P, \quad j \notin P, \quad j \in Succ(i),\\
&\underline{\theta} \leqslant \theta < \bar{\theta},\\
&s_1 = 0,\\
&s_i \geqslant 0, \quad \forall i \in V,
\end{aligned}
\tag{15}
$$

where $\theta = L_{-1}(\lambda),\ \underline{\theta} = L_{-1}(1)$ and $\bar{\theta} = L^{-1}(0)$.

If θ_{min} is the optimal objective value of (15) for θ, then the $Poss(p \text{ is critical}) = L(\theta_{min})$. If the problem is infeasible, then $Poss(p \text{ is critical}) = 0$.

3.2. The degree of possibility that an activity is critical

The following formula determines the possibility that an activity, $k \in V$, is critical [2].

$$Poss(k \text{ is critical}) = \sup_{\Omega: k \text{ is critical in } \Omega} \pi(\Omega) \tag{16}$$

This value can be computed by the possibility distribution of the float as formula (17):

$$Poss(k \text{ is critical}) = \mu_{\tilde{F}_k}(0) \tag{17}$$

The degree of possible criticality of a given activity can be evaluated using path criticalities. Proposition 1 provides a way of determining the index $Poss(k \text{ is critical})$.

Proposition 1. ([4])

$$\mu_{\tilde{F}_k}(0) = \max_{p\in P(k)} \text{Poss}(p \text{ is critical}).$$

Proposition 1 states that the degree of possible criticality of a given activity k, $k \in V$, is the maximum of the degree of possible criticality of all of the paths from P that contain the given activity k. Proposition 1 is the key to constructing an algorithm for computing the degree of possible criticality of activities as a realistic problem on which the literature is completely void.

3.3. The degree of necessity that a path is critical

The necessity that a path is critical is defined as the following:

$$Nec(p \text{ is critical}) = 1 - Poss(p \text{ is not critical}) = \inf_{\Omega: p \text{ is not critical in } \Omega} (1 - \pi(\Omega)) \tag{18}$$

Chanas et al. [2] have proposed two methods for computing the degree of necessity that a path is critical. The first method is based on the concept of bisection and can be adapted to fuzzy activity durations that are given in a general form. The second method is based on linear programming, which can be used for activity durations of fuzzy numbers of the L–L type. Chanas et al. [2] used the activity-on-arc (AOA) convention, whereas the linear programming model in the AOA convention is different from the linear programming model for the activity-on-node (AON) convention. In the following, the linear programming model for this problem in the AON network is presented.

Chanas et al. [2] showed that a path, $p \in P$, is necessarily critical in G^{λ} if and only if the path is a critical path in the usual sense, in the optimistic configuration induced by p, $\underline{\Omega}_p^{\lambda}$. The following model of linear equalities and inequalities ascertained whether the given path, p, is necessarily critical in network G^{λ} for a fixed $\lambda \in (0,1]$.

$$\begin{aligned}
&s_j - s_i - \underline{d}_i^{1-\lambda} = 0 \quad i \in P, \quad j \in P, \quad j \in Succ(i),\\
&s_j - s_i - \underline{d}_i^{1-\lambda} \geqslant 0 \quad i \in P, \quad j \notin P, \quad j \in Succ(i),\\
&s_j - s_i - \bar{d}_i^{1-\lambda} \geqslant 0 \quad i \notin P, \quad j \notin P, \quad j \in Succ(i),\\
&s_1 = 0,\\
&s_i \geqslant 0, \quad \forall i \in V
\end{aligned} \tag{19}$$

Chanas et al. [2] proposed formula (20) to compute the degree of necessity that the given path is critical.

$$Nec(p \text{ is critical}) = \sup\{\lambda | p \text{ is necessarily critical in } \mathrm{G}^{1-\lambda}\} \tag{20}$$

The necessity degree of criticality of the path is calculated by the following programming model.

$$\begin{aligned}
&Maximize \left\{\lambda - \sum_{i \in V} s_i\right\},\\
&s_j - s_i - \underline{d}_i^{1-\lambda} = 0 \quad i \in P, \quad j \in P, \quad j \in Succ(i),\\
&s_j - s_i - \underline{d}_i^{1-\lambda} \geqslant 0 \quad i \in P, \quad j \notin P, \quad j \in Succ(i),\\
&s_j - s_i - \bar{d}_i^{1-\lambda} \geqslant 0 \quad i \notin P, \quad j \notin P, \quad j \in Succ(i),\\
&0 \leqslant \lambda \leqslant 1,\\
&s_1 = 0,\\
&s_i \geqslant 0, \quad \forall i \in V,
\end{aligned} \tag{21}$$

Again, the term $minimize\{\sum_{i \in V} s_i\}$ is added to the objective function. Without this term, the programming model provides the earliest starting times for activities that belong to the given path, and invalid starting times are assigned to other activities. If the problem (21) is infeasible, then $Nec(p \text{ is critical}) = 0$, and if λ_{max} is the optimal object value of (21) for λ, then $Nec(p \text{ is critical}) = \lambda_{max}$.

Assume that fuzzy activity durations $\tilde{d}_i, i \in V$, are given by means of fuzzy numbers of the L–L type. In this case, the $(1-\lambda)$-cuts of a fuzzy number $\tilde{d}_i$ have the following form:

$$\tilde{d}_i^{1-\lambda} = \left[\underline{d}_i^{1-\lambda}, \bar{d}_i^{1-\lambda}\right] = \left[\underline{d}_i - L^{-1}(1-\lambda)\alpha_i, \bar{d}_i + L^{-1}(1-\lambda)\beta_i\right].$$

To transfer the model (21) to the following linear programming problem, set $\theta = L^{-1}(1-\lambda)$.

$$\begin{aligned}
&Maximize \left\{\theta - \sum_{i \in V} s_i\right\},\\
&s_j - s_i - \underline{d}_i + \alpha_i\theta = 0 \quad i \in P, \quad j \in P, \quad j \in Succ(i),\\
&s_j - s_i - \underline{d}_i + \alpha_i\theta \geqslant 0 \quad i \in P, \quad j \notin P, \quad j \in Succ(i),\\
&s_j - s_i - \bar{d}_i - \beta_i\theta \geqslant 0 \quad i \notin P, \quad j \notin P, \quad j \in Succ(i),\\
&\underline{\theta} \leqslant \theta < \bar{\theta},\\
&s_1 = 0,\\
&s_i \geqslant 0, \quad \forall i \in V,
\end{aligned} \tag{22}$$

where $\underline{\theta} = L^{-1}(1), \bar{\theta} = L^{-1}(0)$. If θ_{max} is the optimal objective value of (22), the $Nec(p \text{ is critical}) = 1 - L(\theta_{max})$. If the problem is infeasible, then $Nec(p \text{ is critical}) = 0$.

Linear programming formulations for determining the possibility and necessity degree of criticality of a path in networks with fuzzy activity and time lag durations have been studied by Yakhchali et al. [37,38]. Yakhchali et al. [44] have also proposed algorithms for computing the latest starting times and the maximal floats in a network with imprecise activities and time lag durations.

3.4. The degree of necessity that an activity is critical

The following formulas determine the necessity that an activity k is critical:

$$Nec(k \text{ is critical}) = 1 - \sup_{\Omega: k \text{ is not critical in } \Omega} \pi(\Omega) = 1 - \sup_{v>0} \mu_{\tilde{F}_k^l}(v) \tag{23}$$

An activity becomes possibly critical before becoming necessarily critical. This fact can be stated with the following:

$$\begin{aligned} &Poss(k \text{ is critical}) < 1 \Rightarrow Nec(k \text{ is critical}) = 0 \\ &Nec(k \text{ is critical}) > 0 \Rightarrow Poss(k \text{ is critical}) = 1 \end{aligned} \tag{24}$$

where $k \in V$. The same relation between the possibly critical and necessarily critical paths can be deduced. This concept is illustrated in Fig. 1, which shows various possibility distributions for the float of an activity k, $k \in V$. In Fig. 1a, the possibility degree of criticality of k, $k \in V$, is 0.6; hence, the necessity degree of criticality of k is zero. Although $Poss(k$ is critical$) = 1$ in Fig. 1b, the necessity degree of criticality of k is still zero. The value of index $Nec(k$ is critical$)$, however, becomes 0.6 in Fig. 1c, while the necessity degree of criticality of k is one in Fig. 1d when the activity is critical at different values of λ.

As mentioned above, the degree of possible criticality of an activity can be computed by means of the degree of possible criticality of all of the paths from P that contain the given activity (based on Proposition 1). Unfortunately, the degree of necessary criticality of an activity cannot be computed through a path's criticality. This fact is illustrated by Fig. 2. In Fig. 2, the number that is adjacent to each node represents the corresponding activity duration. It is assumed that the activity durations are fuzzy numbers of the same L–L *type*, where $L(x) = max(0, 1 - x^2)$.

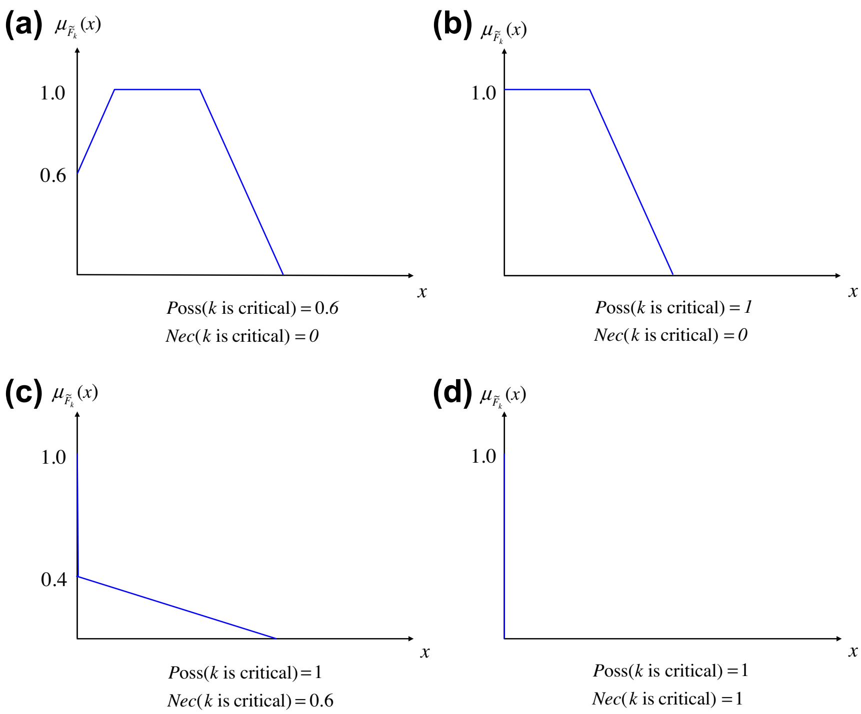

Fig. 1. Various possibility distributions for the float of an activity k, $k \in V$.

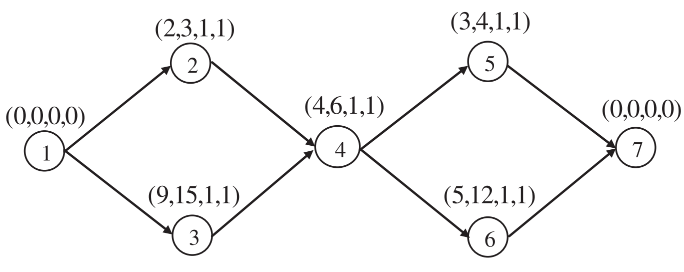

Fig. 2. A counter-example network.

Table 1
The possibility and necessity degrees of criticality of paths in Fig. 2.

Paths	Possibility degree of criticality	Necessity degrees of criticality
1-2-4-5-7	0	0
1-2-4-6-7	0	0
1-3-4-5-7	0.75	0
1-3-4-6-7	1	0.25

The possibility and necessity degrees of criticality of paths in Fig. 2 are listed in Table 1.

In Fig. 2, there are no parallel activities to the activity "4"; therefore, each path leading from "1" to "7" uses "4", which is why Nec(4 is critical) = 1, although the maximum of the degree of necessary criticality for all of the paths that contain "4" is 0.25.

Proposition 2 finds a lower bound for the necessity degrees of criticality of a given activity k, $k \in V$, based on the maximum of the necessity degrees of criticality of the paths from P that contain k.

Proposition 2

$$\max_{p \in P(k)} \mathrm{Nec}(p \text{ is critical}) \leqslant Nec(k \text{ is critical}).$$

Proof 1. Let p' be a path of $P(k)$, $p' \in P(k)$, which maximizes the necessity degrees of the criticality of paths that contain k, as in the following formula:

$$Nec(p' \text{ is critical}) = \max_{p \in P(k)} Nec(p \text{ is critical}) \tag{25}$$

Assume that Ω' is a configuration such that Nec (p' is critical) = $1 - \pi(\Omega')$. Let Ω^* be a configuration for which $Nec(k$ is critical) = $1 - \pi(\Omega^*)$. According to formulas (23) and (18), $\pi(\Omega') \geqslant \pi(\Omega^*)$. It can be deduced that Nec (k is critical) $\geqslant$ $Nec(p'$ is critical). □

$\underline{Nec}(k)$ will denote the lower bound of the necessity degrees of criticality of k, $k \in V$; henceforth, $\underline{Nec}(k) = max_{p \in P(k)}$Nec($p$ is critical).

4. The path enumeration approach for critical analysis

Initially, in the path enumeration approach, all of the paths in the network with fuzzy activity durations are constructed by an algorithm, and the possibility and necessity degrees of criticality of the paths are computed. Then, the possibility degrees of the criticality of activities (based on Proposition 1) and the lower bound of the necessity degrees of the criticality of activities (based on Proposition 2) are evaluated.

4.1. The Path enumeration algorithm

Algorithm 1 computes all of the paths in the network G and determines the possibility degree that corresponds to the criticality of all of the activities and paths and calculates the lower bound of the necessity degree of the criticality of the activities.

Algorithm 1. Determining the degree of possibility that activities are critical

Input: A network $G = \langle V, E \rangle$
Output: The degree of possibility that activities are critical
1: $p \leftarrow \{1\}$
2: call *Enumeration Procedure* (*1*)
3: **for** $i \in V$ **do**
4: $Poss(k$ is critical) = $max\{$Poss (p is critical)| $p \in P(k)\}$
5: **end for**
6: **for** $i \in V$ **do**
7: $\underline{Nec}(k) = max\{Nec(p$ is critical) $p \in P(k)\}$
8: **end for**

Enumeration Procedure (j)

a: **if** $j = n$ **then**
b: Compute the degree of possibility that p is critical; $Poss(p \text{ is critical})$
c: Compute the degree of necessity that p is critical; $Nec(p \text{ is critical})$
d: $Paths \leftarrow Paths \cup \{p\}$
e: **else**
f: **for** $k \in Succ(j)$ **do**
g: $p \leftarrow p \cup \{k\}$
h: call *Enumeration Procedure* (k)
i: **end for**

Paths are made by the recursive '*Enumeration Procedure*'. The call to the procedure in line 2 of Algorithm 1 will construct the set P. The procedure computes the possibility and necessity degrees of criticality of each path as well as saves the current path and its degrees. The possibility degree of criticality of activities is calculated in line 4. Analogously, the algorithm in line 7 computes the lower bound of the necessity degrees of criticality of activities. Determining paths with the maximal necessity degree of criticality is a corollary to the proposed algorithm. Lines 6 to 8 in the algorithm and line "b" in the procedure can be omitted because the aim is to compute only the possibility degree of the criticality of the activities.

This algorithm calculates the degrees of the possible criticality of activities and paths, the degrees of the necessary criticality of paths and the lower bounds of the degrees of the necessary criticality of activities. This strategy is an advantage of the proposed approach compared to previous approaches. Moreover, the algorithm determines paths that have maximum necessity degrees of criticality. In spite of the fact that the problem of computing the possibility degree of criticality of a path has been investigated, its activity counterpart has never been addressed in prior research, and the literature on this real-world problem is completely absent.

4.2. Numerical example

Let us consider a project network in Fig. 3 that was proposed by Chanas and Zielinski [4]. Chanas and Zielinski [4] used the activity-on-arc (AOA) convention, whereas in this paper, the proposed network was converted to the activity-on-node (AON) convention to adopt the paper's network representation. The AOA and AON conventions present a single project; thus, it is possible to use the data that are used in the AOA convention for the AON convention. This conversion is employed to provide a basis for comparing the present approach with previous approaches.

Assume that the activity durations are fuzzy numbers of the same *L–L type*, where $L(x) = max(0, 1 - x^2)$, as follows:

$$\tilde{d}_1 = (0,0,0,0)_{L-L} \quad \tilde{d}_2 = (1,1.5,1,1)_{L-L} \quad \tilde{d}_3 = (2,3,0,2)_{L-L}$$
$$\tilde{d}_4 = (2,3,1,2)_{L-L} \quad \tilde{d}_5 = (9,9,1,1)_{L-L} \quad \tilde{d}_6 = (5,5,1,1)_{L-L}$$
$$\tilde{d}_7 = (6,7,0,2)_{L-L} \quad \tilde{d}_8 = (8,9,2,4)_{L-L} \quad \tilde{d}_9 = (3,4,2,0)_{L-L}$$
$$\tilde{d}_{10} = (4,4,2,2)_{L-L} \quad \tilde{d}_{11} = (6,9,2,3)_{L-L} \quad \tilde{d}_{12} = (0,0,0,0)_{L-L}$$

Algorithm 1 constructs all of the paths for Fig. 3 and computes the possibility and necessity degrees of criticality of paths therein, as listed in Table 2.

The possibility degrees of criticality of activities are calculated through lines 3–5 in Algorithm 1. Eventually, the algorithm based on Proposition 2 computes the lower bounds of the necessity degrees of the criticality of the activities, as listed in Table 3.

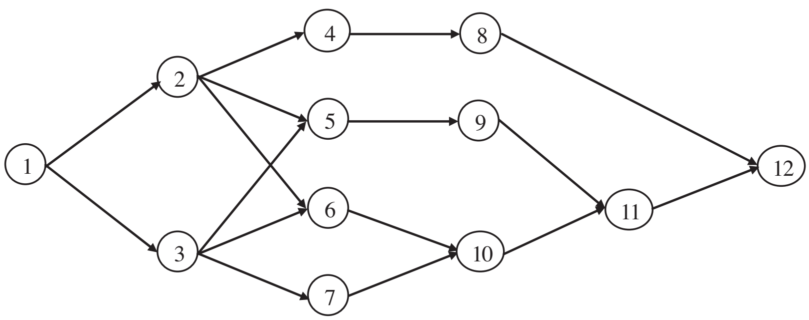

Fig. 3. A project network with *L–L* type fuzzy numbers.

Table 2
The possibility and necessity degrees of criticality of paths in Fig. 3.

Paths	Possibility degree of criticality	Necessity degrees of criticality
1-2-4-8-12	0.7024	0
1-2-5-9-11-12	0.75	0
1-2-6-10-11-12	0.4375	0
1-3-5-9-11-12	1	0.0204
1-3-6-10-11-12	0	0
1-3-7-10-11-12	0.9796	0

Table 3
The possibility degrees and lower bounds of necessity degrees of the criticality of the activities in Fig. 3.

Activities	Possibility degrees of criticality	Lower bounds of necessity degrees of criticality
1	1	0.0204
2	0.75	0
3	1	0.0204
4	0.7024	0
5	1	0.0204
6	0.4375	0
7	0.9796	0
8	0.7024	0
9	1	0.0204
10	0.9796	0
11	1	0.0204
12	1	0.0204

The number of tested paths depends on the topology of the network and is potentially exponential; however, in practice, the algorithm can obtain the results in a reasonable time period on realistic problems. This fact will be discussed next.

4.3. Computational experience

To evaluate the performance of the proposed algorithm, Algorithm 1 was tested on realistic scheduling problems proposed by Kolisch and Sprecher [26]. They presented a set of benchmark instances for project scheduling problems, which have been systematically generated by the standard project generator, ProGen. The work of Kolisch et al. [27] is purported to be representative of real project scheduling problems. The activity durations of those instances are precisely defined; thus, *L–L* type fuzzy numbers were generated based on the crisp activity durations. The choice of those fuzzy numbers is not important for the test because the algorithm complexity only depends on the network topology.

Algorithm 1 calculated the possibility degree of the criticality of activities and paths and the lower bound of the necessity degree of the criticality of activities on project scheduling libraries, which can be downloaded from the PSPLIB web site (http://129.187.106.231/psplib), with 32, 62, 92 and 122 activities on 480, 480, 480 and 600 instances of the problems, respectively. Algorithm 1 has been programmed in MATLAB (R2006b) and was run on a personal computer with 1.60 GHz processor (Intel Centrino 1.7) and 512 MB of RAM. The overall execution times, expressed in seconds, were measured.

Hence, the possibility degrees of criticality of the 158,400 activities in 2040 project networks were computed. The results of Table 4 are demonstrated in Fig. 4 and Fig. 5.

Table 4
The execution times and the number of paths constructed by Algorithm 1.

Nb of activities	32	62	92	122
Nb of networks tested	480	480	480	600
Minimal execution time	0.168	0.3271	0.4761	0.6553
Average execution time	0.6014	1.3701	2.4133	3.7385
Maximal execution time	2.5272	6.3469	11.5661	15.9495
Minimal nb of paths	18	33	48	65
Average nb of paths	56.9792	127.35	221.425	325.6933
Maximal nb of paths	204	563	961	1277
Minimal nb of PRs	48	93	138	183
Average nb of PRs	58	112	166	220
Maximal nb of PRs	68	131	194	257

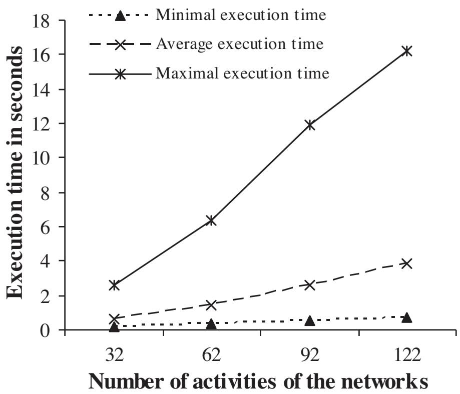

Fig. 4. The execution times of Algorithm 1.

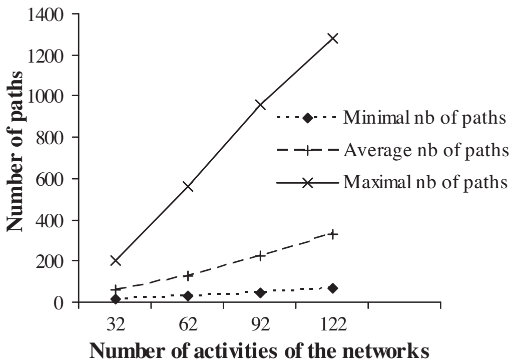

Fig. 5. The number of paths is constructed by Algorithm 1.

These tests show that Algorithm 1 can compute the possibility degrees of criticality of activities in real world project networks in acceptable execution time periods.

5. Conclusions

The project scheduling in networks with fuzzy activity durations is a more realistic method for modeling the problem than traditional approaches with fixed or stochastic activity durations. Intuitively, when the durations are fuzzy numbers, the value of the earliest and latest starting times besides the float of activities become fuzzy numbers as well; hence, there is a lack of robustness in the traditional critical path analysis. Instead of being critical or not, an activity (or a path) is considered critical with a degree of possibility and necessity.

Having presented the linear programming formulations in activity-on-node (AON) networks to evaluate the possibility and necessity degrees of the criticality of paths, a novel approach for determining the degrees of possible and necessary criticality of activities and paths has been proposed. This approach was summarized in an effective algorithm that relies on a path enumeration technique. The proposed algorithm computes the degrees of possible criticality of the activities and paths, the degrees of necessary criticality of the paths and the lower bounds of the degrees of necessary criticality of the activities. In addition, the algorithm determines the paths with the maximum necessity degree of criticality. Although the problem of computing the possibility degree of criticality of a path has been previously studied, the same problem for an activity has never been addressed in prior research, and the literature suffers from a complete void with respect to realistic problems.

The proposed algorithm was tested on real-world project networks, and experimental results showed that the algorithm is fully capable of computing the possibility degrees of criticality of the activities in acceptable time periods. Although it seems possible to further develop some heuristics, e.g., branch and bound, to improve the running time of the algorithm, this approach could be an appropriate topic for further studies.

The proposed algorithm computes lower bounds for the necessity degree of criticality of the activities based on the maximum of the necessity degrees of the criticality of the paths. Providing more conditions that allow for the thorough computation of the necessity degree of the criticality of the activities is another line for future research.

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